Scale-Invariant
Turbine Characteristics

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1 Introduction

Because fluid dynamic equations are difficult to solve, it is common to study the performance of scaled down models.[1] For this technique to be effective, we need to relate the magnitudes occurring in these tests to the corresponding magnitudes in the full scale system. In this note we first review the general principle of dynamic similarity. We then derive certain scale-invariant parameters of dynamically similar turbines. These scale-invariant parameters are used to present model test results in a form which facilitates predicting the performance of geometrically similar turbines operating over a range of pressures and shaft speeds.

2 Similar Systems

2.1 Geometric Similarity

When we speak of scale models we immediately think of mechanical systems with the same shape but different size. More specifically at any instant there is a correspondence between points in system A and points in system B and scale factor s_L such that if x_1 and x_2 are two points in system A and x'_1 and x'_2 are the corresponding points in system B then

$$|x'_2 - x'_1| = s_L\,|x_2 - x_1|\,. \tag{1}$$

In this case we say that system A is *geometrically similar* to system B.

Given the scale factor s_L for lengths, the scale factor for areas will be s_L^2 and the scale factor for volumes will be s_L^3. We say that length, area, and volume are magnitudes with different

D	runner diameter (L)
g	gravitational acceleration (LT^{-2})
H	turbine net head (L)
L	length dimension
M	mass dimension
m	mass (M)
N	shaft speed (RPM)
N_{11}	unit speed
P	power (ML^2T^{-3})
P_{11}	unit power
p	pressure difference across turbine ($ML^{-1}T^{-2}$)
Q	turbine discharge (L^3T^{-1})
Q_{11}	unit discharge
s_L	length scale factor (dimensionless)
s_M	mass scale factor (dimensionless)
s_T	time scale factor (dimensionless)
T	time dimension
t	time (T)
v	velocity (LT^{-1})
$\lvert x_1 - x_2 \rvert$	distance between points x_1 and x_2 (L)
η	efficiency (dimensionless)
Π_1	speed parameter $D\omega\,(p/\rho)^{-1/2}$ (dimensionless)
Π_2	discharge parameter $QD^{-2}\,(p/\rho)^{-1/2}$ (dimensionless)
Π_3	power parameter $P\rho^{-1}D^{-2}\,(p/\rho)^{-3/2}$ (dimensionless)
ρ	fluid density (ML^{-3})
ω	angular shaft velocity (T^{-1})

Table 1: Nomenclature for turbine characteristics.

dimensions and denote these dimensions by L, L^2, and L^3. In discussing scale factors for different dimensions, we assume that the same units are used in both systems to measure magnitudes of the same dimension. No other assumptions are made about the units used for lengths, areas, and volumes. We can measure lengths in feet, areas in acres, and volumes in cubic meters and, given that lengths are scaled by s_L, areas will be scaled by s_L^2 and volumes will be scaled by s_L^3.

2.2 Kinematic Similarity

More generally, the corresponding points of two systems may be functions of time. In this case we say that system A is *kinematically similar* to system B if there are two scale factors s_L and s_T such that for any two points $x_1(t)$ and $x_2(t)$ in system A at time t and corresponding points $x_1'(t')$ and $x_2'(t')$ in system B at the corresponding time t'

$$\begin{aligned} |x_2'(t_1') - x_1'(t_1')| &= s_L |x_2(t_1) - x_1(t_1)| & (2) \\ |x_2'(t_2') - x_1'(t_2')| &= s_L |x_2(t_2) - x_1(t_2)| & (3) \\ |t_2' - t_1'| &= s_T |t_2 - t_1| & (4) \end{aligned}$$

for any two times t_1 and t_2 in system A and corresponding times t_1' and t_2' in system B.

Including time introduces new kinds of magnitudes. For instance, if a particle in system A moves from point x_1 to point x_2 between times t_1 and t_2, it will have an average velocity

$$|v| = \left| \frac{x_2(t_1) - x_1(t_1)}{t_2 - t_1} \right|. \tag{5}$$

The magnitude of the corresponding velocity v' in system B will be

$$\begin{aligned} |v'| &= \left| \frac{x_2'(t_1') - x_1'(t_1')}{t_2' - t_1'} \right| & (6) \\ &= \frac{s_L |x_2(t_1) - x_1(t_1)|}{s_T |t_2 - t_1|} & (7) \\ &= s_L s_T^{-1} \left| \frac{x_2(t_1) - x_1(t_1)}{t_2 - t_1} \right| & (8) \\ &= s_L s_T^{-1} |v|. & (9) \end{aligned}$$

Again, we assume the same units are used in both systems to measure magnitudes of the same dimension. The scale factor for velocities reflects the fact that the measured velocity is proportional to the distance and inversely proportional to the time interval. We say that velocity has dimension LT^{-1}. In a similar fashion, acceleration has dimension LT^{-2} and the scale factor for accelerations is $s_L s_T^{-2}$.

2.3 Dynamic Similarity

An additional basic dimension is required to complete the classification of magnitudes in mechanical systems. The third basic dimension can be either force or mass. Here we choose dimension mass M. Kinematically similar systems A and B (with length and time scale factors s_L and s_T) are also *dynamically similar* if there is a fixed scale factor for mass s_M relating corresponding masses. Specifically, if m is the mass contained in a certain volume V at time t in system A , then the mass m' contained in the corresponding volume V' at the corresponding time t' in system B are related by

$$m' = s_M m. \tag{10}$$

We refer to magnitudes with dimensions derived from L, M, and T as dynamic magnitudes. Three examples with their dimensions and scale factors are

1. density ML^{-3}; $s_M s_L^{-3}$,
2. force MLT^{-2}; $s_M s_L s_T^{-2}$, and
3. pressure $ML^{-1}T^{-2}$; $s_M s_L^{-1} s_T^{-2}$.

2.4 Creation of Dynamically Similar Systems

Suppose all the dynamic magnitudes in system A are determined by certain given or independent magnitudes (under our control) and certain physical laws. We then create a new system B by scaling the independent magnitudes of system A according to their dimension and a set of scale factors s_L, s_M, and s_T. If system B is governed by the same physical laws, then system B will be dynamically similar to system A. In particular, if system A is a working model in which we can measure various magnitudes, then we can calculate the corresponding magnitudes in system B by scaling the measured magnitudes in accordance with their dimensions and the same scale factors s_L, s_M, and s_T.[2]

3 Turbine Model Testing

To see how dynamic similarity applies to turbine characteristics, consider the case in which a model turbine operating with

- fluid density ρ (dimension ML^{-3}),
- pressure difference p (dimension $ML^{-1}T^{-2}$), and
- shaft speed ω (dimension T^{-1})

has a resulting

- discharge Q (dimension L^3T^{-1}) and
- shaft power P (dimension ML^2T^{-3}).

For any choice of positive scale factors s_M, s_L, and s_T, we know that a prototype turbine operating with

- fluid density $s_M s_L^{-3} \rho$,

- pressure difference $s_M s_L^{-1} s_T^{-2} p$, and
- shaft speed $s_T^{-1} \omega$

will have a resulting

- discharge $s_L^3 s_T^{-1} Q$ and
- shaft power $s_M s_L^2 s_T^{-3} P$.

4 Scale-Independent Turbine Characteristics

There remains a practical problem. Given a geometrically similar prototype with

1. runner diameter D_2,
2. fluid density ρ_2,
3. pressure difference p_2, and
4. angular shaft speed ω_2,

how do we locate model test data with

1. runner diameter D_1,
2. fluid density ρ_1,
3. pressure difference p_1, and
4. shaft speed ω_1

so that the prototype discharge Q_2 and power P_2 can be computed from the model discharge Q_1 and model power P_1 using the principle of dynamic similarity?

For such a computation, we need three positive scale factors s_L, s_M, and s_T such that

$$
\begin{aligned}
D_2 &= s_L D_1 && (11) \\
\rho_2 &= s_M s_L^{-3} \rho_1 && (12) \\
p_2 &= s_M s_L^{-1} s_T^{-2} p_1 && (13) \\
\omega_2 &= s_T^{-1} \omega_1. && (14)
\end{aligned}
$$

The first three equations are consistent with only one choice of scale factors

$$s_L = \left(\frac{D_2}{D_1}\right) \tag{15}$$

$$s_M = \left(\frac{D_2}{D_1}\right)^3 \left(\frac{\rho_2}{\rho_1}\right) \tag{16}$$

$$s_T = \left(\frac{D_2}{D_1}\right) \left(\frac{\rho_2}{\rho_1}\right)^{1/2} \left(\frac{p_2}{p_1}\right)^{-1/2} \tag{17}$$

for length, mass, and time. All four equations are satisfied with this choice of scale factors if and only if

$$\frac{\omega_2}{\omega_1} = s_T^{-1} = \left(\frac{D_2}{D_1}\right)^{-1} \left(\frac{\rho_2}{\rho_1}\right)^{-1/2} \left(\frac{p_2}{p_1}\right)^{1/2}. \tag{18}$$

And, provided this is the case, the operation of the model and prototype turbines will be dynamically similar and we will have

$$Q_2 = s_L^3 s_T^{-1} Q_1 \tag{19}$$

and

$$P_2 = s_M s_L^2 s_T^{-3} P_1. \tag{20}$$

Substituting the expressions for s_L, s_M, and s_T (Equations 15, 16, and 17) into Equations 18, 19, and 20 and rearranging, we obtain the following: If

$$D_2 \rho_2^{1/2} p_2^{-1/2} \omega_2 = D_1 \rho_1^{1/2} p_1^{-1/2} \omega_1 \tag{21}$$

then

$$D_2^{-2} \rho_2^{1/2} p_2^{-1/2} Q_2 = D_1^{-2} \rho_1^{1/2} p_1^{-1/2} Q_1 \tag{22}$$

and

$$D_2^{-2} \rho_2^{1/2} p_2^{-3/2} P_2 = D_1^{-2} \rho_1^{1/2} p_1^{-3/2} P_1. \tag{23}$$

The left and right sides of the three equations above are scale-invariant parameters for dynamically similar turbines.[3] We will refer to them as the *speed parameter*

$$\Pi_1 = \frac{D\omega}{(p/\rho)^{1/2}}, \tag{24}$$

the *discharge parameter*

$$\Pi_2 = \frac{Q}{D^2 (p/\rho)^{1/2}}, \tag{25}$$

and the *power parameter*

$$\Pi_3 = \frac{P}{\rho D^2 (p/\rho)^{3/2}}. \tag{26}$$

If the model and prototype are operating under conditions producing the same speed parameter Π_1, then they will be dynamically similar and will have the same discharge parameter

Π_2 and power parameter Π_3. Based on this principle, model test data is commonly summarized by two curves: Π_2 versus Π_1 and Π_3 versus Π_1. To calculate the discharge and power of a prototype, we

1. compute the speed parameter Π_1 of the prototype based on its runner diameter D_2, angular speed ω_2, pressure difference p_2, and fluid density ρ_2,
2. look up the values of Π_2 and Π_3 for the same speed parameter Π_1 in the model test data, and
3. compute the prototype discharge

   $$Q_2 = \Pi_2 D_2^2 \left(p_2/\rho_2\right)^{1/2} \tag{27}$$

   and power

   $$P_2 = \Pi_3 \rho_2 D_2^2 \left(p_2/\rho_2\right)^{3/2}. \tag{28}$$

For turbines with adjustable gates, changing the gate position changes the turbine shape. The principle of dynamic similarity only applies to geometrically similar machines. Thus we have different curves of Π_2 and Π_3 for different gate positions.

5 Alternate Formulations

It is common to plot the efficiency

$$\eta = \frac{\Pi_3}{\Pi_2} = \frac{P}{pQ} \tag{29}$$

versus the speed parameter Π_1 instead of the power parameter Π_3 versus the speed parameter Π_1. In the hydro industry, shaft speed N is commonly expressed in RPM and pressure p is expressed in terms of the height H of a column of water such that

$$p = \rho g H. \tag{30}$$

With these conventions, the scale invariant parameters become

$$\Pi_1 = \frac{DN}{\left(gH\right)^{1/2}} \tag{31}$$

$$\Pi_2 = \frac{Q}{D^2 \left(gH\right)^{1/2}} \tag{32}$$

$$\Pi_3 = \frac{P}{\rho D^2 \left(gH\right)^{3/2}}. \tag{33}$$

In the case of hydro turbines where the fluid is water for both model and prototype, the acceleration of gravity g and fluid density ρ are nearly constant. Therefore these factors are routinely omitted to obtain simplified scale-invariant parameters

$$N_{11} = \frac{DN}{H^{1/2}} \tag{34}$$

$$Q_{11} = \frac{Q}{D^2 H^{1/2}} \tag{35}$$

$$P_{11} = \frac{P}{D^2 H^{3/2}}. \tag{36}$$

The units of these parameters are somewhat confusing. They are commonly given as RPM, ft^3/s or m^3/s, and horsepower or kW respectively. The equations do not have these units. The parameters are described as the values of speed, discharge, and power for a dynamically similar turbine with unit diameter $D = 1$ operating under unit head $H = 1$. This interpretation is signified by the subscript 11 and they are referred to as *unit parameters*. Of course the numeric values of the parameters depend on whether $H = 1$ means one foot or one meter. The virtue of the unit parameters is that they eliminate a lot of arithmetic in relating model data to prototype operation. Here is the procedure to calculate the discharge and power for a prototype turbine with runner diameter D turning at RPM N under a head H.

1. Calculate $N_{11} = DN\sqrt{H}$.
2. Look up Q_{11} and P_{11} from model data presented as graphs of Q_{11} and P_{11} versus N_{11}.
3. Calculate discharge $Q = D^2 H^{1/2} Q_{11}$.
4. Calculate power $P = D^2 H^{3/2} P_{11}$.

References

[1] John Smeaton, "An experimental Enquiry Concerning the Natural Powers of Water and Wind to Turn Mills," Philos. Trans. R. Soc. London, 1760.

[2] E. Buckingham, "On Physically Similar Systems; Illustrations of the Use of Dimensional Equations," Phys. Rev. **4**: 345-376, 1914.

[3] William W. Peng, Fundamentals of Turbomachinery, Wiley, 2007.