Modeling Hydraulic Transients Using Dynamic System Simulation Software

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Contents

1 Introduction

In this report we derive the continuity and momentum equations applicable to one-dimensional fluid transients in uniform pipe sections without friction loss. We then show how the method of characteristics provides a general solution to the continuity and momentum equations in the form of two waves traveling in opposite directions. Finally we present a library of standard program blocks which can be combined to form a complete fluid transient simulation program. This library contains program blocks which model

1. elementary pipe sections each having uniform diameter, uniform wall stiffness, and no losses, and
2. junctions between elementary pipe sections.

Traveling in opposite directions within an elementary pipe section, the two waves are each modeled by an ideal delay element having a delay time equal to the wave travel time through the elementary pipe section. The initial wave magnitudes entering each end of an elementary pipe segment (i.e. boundary conditions) are determined by the junctions between the elementary pipe sections.

The junctions between elementary pipe sections include hydraulic head losses proportional to the square of the fluid velocity. Pipes with losses are modeled by a sequence of elementary sections connected by junctions having appropriate loss coefficients to account for the pipe hydraulic head losses. Loss junctions can also be added to account for form losses such as pipe bends. Valves and turbines are modeled by junctions with variable loss coefficients. The reservoir and discharge ponds are modeled as limiting cases of junctions between an ordinary elementary pipe section and an infinite diameter elementary section in which the fluid velocity is zero.

2 Terminology

2.1 Nomenclature

The nomenclature used in this report is given in Table 1 (page 2) along with the International System (SI) units for each variable.

A	cross-sectional area of pipe (m^2)
A_d	cross-sectional area of draft tube exit (m^2)
a	wave speed (m/ s)
C_V	flow coefficient (m^3/ s @ 1 m head)
D	inside diameter of pipe (m)
E	Young modulus (Pa)
e	pipe wall thickness (m)
F_j	H_- at downstream ($j = 1$) or upstream ($j = 2$) side of component (m)
f	Darcy-Weisbach friction factor (dimensionless)
f_j	H_+ at downstream ($j = 1$) or upstream ($j = 2$) side of component (m)
G	shear modulus (Pa)
g	gravitational acceleration (m/ s^2)
H	head (m)
K_A	pipe area modulus (Pa)
K_μ	linear density modulus (Pa)
K_ρ	fluid density modulus (Pa)
k	loss coefficient (dimensionless)
L	pipe section length (m)
P	shaft power (W)
p	static pressure (Pa)
p_0	thermodynamic pressure (Pa)
p_1	bulk viscous pressure (Pa)
p_a	atmospheric pressure (Pa)
Q	fluid flow rate (m^3/ s)
T	pipe section travel time (s)
t	time (s)
V	fluid velocity (m/ s)
x	distance along pipe, positive direction towards plant (m)
z	elevation (m)
z_0	intake surface elevation (m)
z_d	discharge surface elevation (m)
γ	fluid specific weight (N/ m^3)
η	turbine efficiency (dimensionless)
μ	linear mass density (kg/ m)
ρ	fluid density (kg/ m^3)
ν	Poisson's ratio for pipe wall (dimensionless)

Table 1: Nomenclature for hydraulic transient analysis.

2.2 Terms used to Describe Pressure and Head

In hydraulics and fluid mechanics, various kinds of *pressure* and *head* are distinguished by the following adjectives:

1. static,
2. hydrostatic,
3. piezometric,
4. velocity or dynamic, and
5. total.

The first two adjectives listed above are particularly confusing since there is a tendency to use the terms "static pressure" and "hydrostatic pressure" interchangeably. In hydro engineering it is also common to use the terms "pressure" and "static pressure" where traditional fluid mechanics texts use "static pressure" and "hydrostatic pressure." The other three kinds of pressure (or head) are not physical pressures but are potential energy densities having dimensions of force per unit area (or length). To fix the terminology used in this document we provide below definitions of the various physical pressures and potential energy densities.

2.3 Physical Pressures

Static pressure p is the sum of the thermodynamic pressure p_0 and the bulk viscous pressure p_1. The bulk viscous pressure p_1 is generally only significant on a cosmological scale or in regions where a fluid is undergoing very rapid expansion or contraction. For the cases considered in this document, the static pressure p is essentially equal to the thermodynamic pressure p_0.

Hydrostatic pressure is the thermodynamic pressure within a connected fluid system which

1. extends to the upper limits of the atmosphere and
2. is in static equilibrium with the gravitational field.

With these definitions, hydrostatic pressure applies only to particular fluid systems at rest with respect to the planet and static pressure is defined for all fluids in all states of motion.

2.4 Potential Energy Densities

The *potential mechanical energy density* (potential mechanical energy per unit volume of fluid) for an incompressible fluid is

$$\rho g z + p + \frac{1}{2}\rho V^2 \tag{1}$$

where ρ is the fluid mass density (mass per unit volume), g is the gravitational acceleration, z is the elevation, p is the static pressure, and V is the fluid velocity. The total potential energy density is the sum of the *gravitational potential energy density* $\rho g z$, the static pressure p, and the *dynamic pressure* $\rho V^2/2$. Since we are only interested in differences in potential energy, the elevation z can be the height above any fixed equipotential surface. Likewise the static pressure p can be relative to any fixed reference pressure. It is common to use elevations relative to mean sea level and pressures relative to atmospheric pressure.

The fluid *specific weight* or gravitational force per unit volume is

$$\gamma = \rho g. \tag{2}$$

The *potential mechanical energy per unit weight of fluid* or *total head* is

$$\frac{p + \rho g z + \frac{1}{2}\rho V^2}{\gamma} = z + \frac{p}{\gamma} + \frac{V^2}{2g}. \tag{3}$$

The total head is the sum of the elevation z, the *static* or *pressure head* p/γ, and the *velocity* or *dynamic head* $V^2/(2g)$. The sum of the first two terms $z + p/\gamma$ is the *piezometric* or *hydraulic head*. When plotting the total head and hydraulic head versus distance along a pipe, the two curves are referred to as the *energy grade line* and the *hydraulic grade line* respectively.

The *gross head* of a hydro plant is the difference between the intake and discharge surface elevations $z_0 - z_d$. The *net head of a turbine* is the difference ΔH between the total head at the turbine inlet and the total head at the turbine discharge. The *turbine efficiency* is

$$\eta = \frac{P}{\gamma Q \Delta H} \tag{4}$$

where P is the shaft power and Q is the turbine discharge (volume per unit time). By convention, the *total head at the discharge of a Francis turbine* is defined to be

$$z_d + p_a + \frac{1}{2g}\left(\frac{Q}{A_d}\right)^2 \tag{5}$$

where z_d is the discharge surface elevation, p_a is atmospheric pressure, and A_d is the cross-sectional area of the draft tube exit.

3 Derivation of Hydraulic Transient Equations

3.1 Derivatives Along a Path

The basic transient equations are more intuitive when written in terms of derivatives along paths following the fluid flow. And in the method of characteristics we solve the transient equations by means of derivatives along paths following the pressure waves. Here we review the concept of a "derivative along a path" in the simple case of one-dimensional motion.

Suppose f is a function of position x and time t with continuous partial derivatives, and c is a curve or path determining x as a function of t

$$x = c(t) \tag{6}$$

with a continuous derivative $c'(t)$. Then "along the path c" the function f can be viewed as a function of the single variable t according to

$$f(t) = f(c(t), t). \tag{7}$$

The derivative of f along the path c is then

$$f'(t) = \frac{\partial f}{\partial x}(c(t), t)\, c'(t) + \frac{\partial f}{\partial t}(c(t), t). \tag{8}$$

This can be written more succinctly as

$$\frac{df}{dt} = \frac{\partial f}{\partial x} c' + \frac{\partial f}{\partial t} \tag{9}$$

or

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial t}. \tag{10}$$

Paths along which the rate of change in position c' is the same as the fluid velocity V

$$c'(t) = V(c(t), t) \tag{11}$$

have a special significance in fluid mechanics equations. These paths "go with the flow" and each such path is the path of a "fluid particle." The derivative of a function f along such a path

$$\frac{df}{dt} = \frac{\partial f}{\partial x} V + \frac{\partial f}{\partial t} \tag{12}$$

is called a *total derivative*. To emphasize the special role of these derivatives, we use the notation

$$\frac{Df}{Dt} = \frac{\partial f}{\partial x} V + \frac{\partial f}{\partial t}. \tag{13}$$

3.2 Basic Variables

In practical problems there are typically a number of pipe sections with each section having uniform cross-sectional area A. More precisely, we assume that the cross-sectional area is constant for the same internal pressure. In developing the equations describing fluid flow through such a pipe section, the following quantities are considered to be functions of distance x along the pipe and time t:

1. A cross-sectional area of pipe,
2. ρ fluid density,
3. V fluid velocity (positive for movement in the positive x direction), and
4. p static pressure.

The percent changes in the fluid density ρ and the cross-sectional area of pipe A are very small over the range of realistic static pressures p. Nevertheless these very small percent changes have a significant effect in the hydraulic transients occurring in long pipes.

We will assume that the four basic functions above are differentiable with respect to x and t. We will also assume that all partial derivatives appearing in this work are continuous so that we can employ the chain and product rules when evaluating derivatives.

3.3 Continuity of Mass

We define the *linear mass density* (mass per unit length) to be

$$\mu(x,t) = \rho(x,t)\, A(x,t)\,. \tag{14}$$

The mass m contained in the interval $[x_1, x_2]$ at time t is

$$\int_{x_1}^{x_2} \mu(x,t)\; dx \tag{15}$$

and the rate of change in mass in the same interval is

$$\frac{d}{dt}\int_{x_1}^{x_2} \mu(x,t)\; dx = \int_{x_1}^{x_2} \frac{\partial \mu}{\partial t}(x,t)\; dx. \tag{16}$$

The *mass flow rate*, or rate at which mass is flowing (past a given point x at a given time t), is the function μV defined by

$$\mu V(x,t) = \mu(x,t)\, V(x,t)\,. \tag{17}$$

If mass is neither created nor destroyed, then the rate of change in mass in the interval $[x_1, x_2]$ is also the difference between the mass flow rate μV at the two end points

$$\frac{d}{dt}\int_{x_1}^{x_2} \mu(x,t)\ dx = \mu V(x,t)\big|_{x=x_2}^{x=x_1} = -\int_{x_1}^{x_2} \frac{\partial \mu V}{\partial x}(x,t)\ dx. \tag{18}$$

From Equations 16 and 18, it follows that

$$0 = \int_{x_1}^{x_2} \frac{\partial \mu}{\partial t}(x,t)\ dx + \int_{x_1}^{x_2} \frac{\partial \mu V}{\partial x}(x,t)\ dx = \int_{x_1}^{x_2} \frac{\partial \mu}{\partial t}(x,t) + \frac{\partial \mu V}{\partial x}(x,t)\ dx. \tag{19}$$

But the only way an integral of a function can be zero on every interval $[x_1, x_2]$ is for the function to be identically zero for all x. From continuity of mass, we must then have

$$0 = \frac{\partial \mu}{\partial t}(x,t) + \frac{\partial \mu V}{\partial x}(x,t)\,. \tag{20}$$

And, if we keep in mind that a partial derivative of a function of x and t is also a function of x and t, we can write this more succinctly as

$$0 = \frac{\partial \mu}{\partial t} + \frac{\partial \mu V}{\partial x}. \tag{21}$$

3.4 Fluid Momentum

The product μV is also the *linear momentum density* (linear momentum per unit length). The linear momentum contained in the interval $[x_1, x_2]$ at time t is

$$\int_{x_1}^{x_2} \mu V(x,t)\ dx \tag{22}$$

and the rate of change in linear momentum in the same interval is

$$\frac{d}{dt}\int_{x_1}^{x_2} \mu V(x,t)\ dx = \int_{x_1}^{x_2} \frac{\partial \mu V}{\partial t}(x,t)\ dx. \tag{23}$$

The *linear momentum flow rate*, or rate at which linear momentum is flowing (past a given point x at a given time t), is given by the function μV^2 defined by

$$\mu V^2(x,t) = \mu V(x,t)\, V(x,t)\,. \tag{24}$$

We assume that the only forces acting on the fluid in the x direction are the static pressure p acting on the cross sectional area A and the the force of gravity due to changes in elevation z of the pipe. The strong form of Newton's Second Law requires that the rate of change in linear momentum within the interval $[x_1, x_2]$ is the sum of the net linear momentum inflow rate plus the net force. We thus have

$$\begin{aligned} \frac{d}{dt}\int_{x_1}^{x_2} \mu V(x,t)\, dx &= \mu V^2(x,t)\Big|_{x=x_2}^{x=x_1} + Ap(x,t)\Big|_{x=x_2}^{x=x_1} + \mu g z(x)\Big|_{x=x_2}^{x=x_1} \\ &= -\int_{x_1}^{x_2} \frac{\partial \mu V^2}{\partial x}(x,t) + A\frac{\partial p}{\partial x}(x,t) + \mu g \frac{dz}{dx}\, dx \end{aligned} \tag{25}$$

where the pipe cross-sectional area A and the linear mass density μ in the second and third terms of the integral are nominal values for the pipe section. From Equations 23 and 25, it follows that

$$0 = \int_{x_1}^{x_2} \frac{\partial \mu V}{\partial t}(x,t) + \frac{\partial \mu V^2}{\partial x}(x,t) + A\frac{\partial p}{\partial x}(x,t) + \mu g\frac{dz}{dx}(x)\, dx. \tag{26}$$

But the only way an integral of a function can be zero on every interval $[x_1, x_2]$ is for the function to be identically zero for all x. From the law of momentum, we must then have

$$0 = \frac{\partial \mu V}{\partial t} + \frac{\partial \mu V^2}{\partial x} + A\frac{\partial p}{\partial x} + \mu g\frac{dz}{dx}. \tag{27}$$

We will refer to Equations 21 and 27 as the *continuity* and *momentum* equations respectively.

3.5 Continuity and Momentum Equations Along a Fluid Particle Path

Expanding Equation 21 using the product rule, we obtain the equivalent continuity equation

$$0 = \frac{1}{\mu}\frac{D\mu}{Dt} + \frac{\partial V}{\partial x}. \tag{28}$$

Expanding Equation 28 using the product rule, we find that two terms can be eliminated on account of the continuity equation. We are then left with

$$0 = \mu\frac{DV}{Dt} + A\frac{\partial p}{\partial x} + \mu g\frac{dz}{dx} \tag{29}$$

or equivalently

$$0 = \rho\frac{DV}{Dt} + \frac{\partial p}{\partial x} + \rho g\frac{dz}{dx}. \tag{30}$$

3.6 Relationship Between Linear Mass Density and Pressure

Within a uniform pipe section, the linear mass density μ increases with static pressure p according to

$$\frac{1}{\mu}\frac{d\mu}{dp} = \frac{1}{\rho}\frac{d\rho}{dp} + \frac{1}{A}\frac{dA}{dp} = \frac{1}{K_\rho} + \frac{1}{K_A} = \frac{1}{K_\mu} \tag{31}$$

where

$$K_\rho = \frac{dp}{d\rho/\rho} \tag{32}$$

is the *density (or bulk) modulus* of the fluid,

$$K_A = \frac{dp}{dA/A} \tag{33}$$

is the *area modulus* of the pipe, and

$$K_\mu = \frac{dp}{d\mu/\mu} = \left(\frac{1}{K_\rho} + \frac{1}{K_A}\right)^{-1} \tag{34}$$

is the *linear mass density modulus.* Using Equation 31, we can rewrite Equation 28 as

$$0 = \frac{1}{\mu}\frac{d\mu}{dp}\frac{Dp}{Dt} + \frac{\partial V}{\partial x} = \frac{1}{K_\mu}\frac{Dp}{Dt} + \frac{\partial V}{\partial x} \tag{35}$$

or, more simply,

$$0 = \frac{Dp}{Dt} + K_\mu\frac{\partial V}{\partial x}. \tag{36}$$

4 Solution by Method of Characteristics

We now consider how to solve for the static pressure p and fluid velocity V in a uniform frictionless pipe. We begin by rewriting Equations 36 and 30 in terms of partial derivatives to obtain the continuity and momentum equations

$$0 = \frac{\partial p}{\partial t} + V\frac{\partial p}{\partial x} + K_\mu\frac{\partial V}{\partial x} \tag{37}$$

and

$$0 = \rho\frac{\partial V}{\partial t} + \rho V\frac{\partial V}{\partial x} + \frac{\partial p}{\partial x} + \rho g\frac{dz}{dx}. \tag{38}$$

In these equations, the unknown static pressure p and fluid velocity V are functions of the two independent variables, position x and time t. Likewise, the partial derivatives of the static pressure p and the fluid velocity V are functions of the same two independent variables. In the method of characteristics we seek curves c along which partial derivatives, such as those appearing in Equations 37 and 38, are reduced to ordinary derivatives. Such curves are called *characteristic curves.* In seeking characteristic curves for the system of Equations 37 and 38 it is helpful to remember that any two distinct linear combinations of these two equations of the form

$$0=\left[\frac{\partial p}{\partial t}+V\frac{\partial p}{\partial x}+K_\mu\frac{\partial V}{\partial x}\right]+\lambda\left[\rho\frac{\partial V}{\partial t}+\rho V\frac{\partial V}{\partial x}+\frac{\partial p}{\partial x}+\rho g\frac{dz}{dx}\right] \tag{39}$$

will constitute a system of partial differential equations having the same solutions as the original Equations 37 and 38. Rewriting the general linear combination above in the form

$$0=\left[(V+\lambda)\frac{\partial p}{\partial x}+\frac{\partial p}{\partial t}\right]+\lambda\rho\left[\left(V+\frac{K_\mu}{\rho\lambda}\right)\frac{\partial V}{\partial x}+\frac{\partial V}{\partial t}+g\frac{dz}{dx}\right] \tag{40}$$

we see that we can reduce the linear combinations of partial derivatives to ordinary derivatives along a curve c if

$$c'(t)=V+\lambda=V+\frac{K_\mu}{\rho\lambda}. \tag{41}$$

The two distinct values of λ satisfying

$$V+\lambda=V+\frac{K_\mu}{\rho\lambda} \tag{42}$$

are

$$\lambda=\pm\sqrt{\frac{K_\mu}{\rho}}. \tag{43}$$

Corresponding to these two distinct values of λ are two requirements for characteristic curves

$$c'(t)=V\pm\sqrt{\frac{K_\mu}{\rho}}=V\pm a \tag{44}$$

where

$$a=\sqrt{\frac{K_\mu}{\rho}} \tag{45}$$

is called the *wave speed* for reasons which will become clear shortly. Along these two characteristic curves, Equation 40 reduces to two ordinary differential equations

$$0=\frac{dp}{dt}\pm\rho a\left(\frac{dV}{dt}+g\frac{dz}{dx}\right). \tag{46}$$

With these two ordinary differential equations we can begin to see why pressure transients are a wave phenomenon. For instance, consider the case of a level pipe section where $dz/dx=0$. We then have the two ordinary differential equations

$$0=\frac{dp}{dt}\pm\rho a\frac{dV}{dt}=\frac{d}{dt}\left(p\pm\rho aV\right). \tag{47}$$

The two quantities

$$p_+ = \frac{p + \rho a V}{2} \quad \text{and} \quad p_- = \frac{p - \rho a V}{2} \tag{48}$$

are not generally constant along a pipe section. Along a path with velocity $V + a$, however, p_+ is constant since

$$\frac{dp_+}{dt} = \frac{1}{2}\frac{d}{dt}\left(p + \rho a V\right) = 0. \tag{49}$$

For this reason we say that p_+ is a "pressure wave" propagating with velocity $V + a$. By similar reasoning, p_- is a pressure wave propagating with velocity $V - a$. The propagation velocities relative to the fluid are $+a$ and $-a$. Thus the wave speed a is the speed at which the pressure waves propagate through the fluid.

Passing through a given point x at a given time t, will be one path with velocity $V + a$ along which p_+ is constant and a second path with velocity $V - a$ along which p_- is constant. It follows that at the given point x at the given time t we will have

$$p_+ + p_- = \frac{p + \rho a V}{2} + \frac{p - \rho a V}{2} = p. \tag{50}$$

We summarize this by saying that the pressure p at any point and time is the sum of two pressure waves propagating at velocities $V + a$ and $V - a$.

For hydro plants, the wave speed a is typically about $1000\,\mathrm{m/\,s}$ and the maximum fluid speed $|V|$ is typically less than $3\,\mathrm{m/\,s}$. In these cases the propagation velocities $V \pm a$ are nearly the same as the relative propagation velocities $\pm a$ and we approximate the requirements for the characteristic curves by

$$c'(t) = \frac{dx}{dt} = \pm a. \tag{51}$$

With this approximation Equation 46 becomes

$$\begin{aligned} 0 &= \frac{dp}{dt} \pm \rho a \frac{dV}{dt} + \rho g \frac{dx}{dt}\frac{dz}{dx} \\ &= \frac{dp}{dt} \pm \rho a \frac{dV}{dt} + \rho g \frac{dz}{dt} \\ &= \frac{d}{dt}\left(p + \rho g z \pm \rho a V\right). \end{aligned} \tag{52}$$

Dividing by $\gamma = \rho g$, we then have

$$0 = \frac{d}{dt}\left(\frac{p}{\gamma} + z \pm \frac{a}{g}V\right) = \frac{d}{dt}\left(H \pm \frac{a}{g}V\right) \tag{53}$$

where

$$H = \frac{p}{\gamma} + z \tag{54}$$

is the hydraulic head.

Defining the two quantities

$$H_+ = \frac{1}{2}\left(H + \frac{a}{g}V\right) \quad \text{and} \quad H_- = \frac{1}{2}\left(H - \frac{a}{g}V\right) \tag{55}$$

we see that H_+ is constant along any path with $dx/dt = a$ and that H_- is constant along any path with $dx/dt = -a$. We call H_+ and H_- *head waves*. And at the intersection of two paths having velocities $dx/dt = \pm a$, we will have hydraulic head

$$H = H_+ + H_- \tag{56}$$

and fluid velocity

$$V = \frac{g}{a}\left(H_+ - H_-\right). \tag{57}$$

5 Library of Standard Program Blocks

5.1 Program Block Conventions

We adopt the conventions that the positive x direction is "downstream" towards the plant, static pressure p is relative to atmospheric pressure, and elevation z is with respect to sea level. At $t = 0$ when the program starts, the hydraulic heads in all pipe sections from the inlet to the first closed valve are initialized to the intake surface elevation z_0. The initial hydraulic heads in sections between the discharge and the first closed valve are initialized to the discharge surface elevation z_d.

The standard program blocks model two kinds of components: pipe sections and junctions between pipe sections. Each of these components has an upstream side (closest to the intake) and a downstream side (closest to the discharge). The graphical representation of each standard program block likewise has an upstream and downstream side corresponding to the sides of the component which it models. Quantities pertaining to the downstream side of a block (closest to the discharge) are subscripted with 1 and quantities pertaining to the upstream side of a component (closest to the intake) are subscripted with 2. The $H_\pm$ waves are represented explicitly by time-dependent inputs and outputs of the program blocks. The block inputs and outputs associated with the two head waves H_+ and H_- are labelled as follows:

1. output f_1: the H_+ wave exiting the downstream side,
2. input f_2: the H_+ wave entering the upstream side,

3. input F_1: the H_- wave entering the downstream side, and
4. output F_2: the H_- wave exiting the upstream side.

Details of the simulation program components are described below, beginning with the uniform frictionless pipe section.

5.2 W-0: Elementary Pipe Section

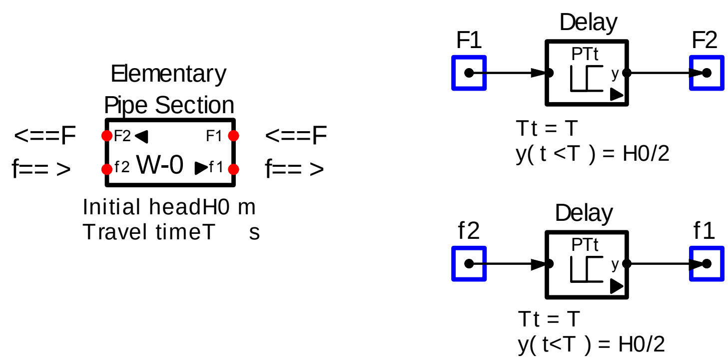

Figure 1: Block diagram of pipe section with no friction loss.

Figure 1 shows the program block W-0 for an elementary pipe section. The appearance of the block when inserted in a larger program is shown on the left and the block diagram is shown on the right. Positive x direction toward plant is from side 2 (left) to side 1 (right). The block has two parameters:

1. travel time T and
2. initial hydraulic head H_0.

The travel time T for the pipe section is

$$T = \frac{L}{a} \tag{58}$$

where L is the pipe section length and a is the pipe section wave speed. The initial head H_0 is the hydraulic head at $t = 0$ when the program begins. The input F_1 at time t (the H_- wave entering side 1 at time t) appears at the output F_2 at time $t+T$. Likewise, the input f_2 at time t (the H_+ wave entering side 2 at time t) appears at the output f_1 at time $t+T$.

5.3 Junctions Between Dissimilar Pipe Sections

5.3.1 Loss coefficients between dissimilar pipe sections

In a junction between dissimilar pipe sections, the sections may have unequal wave speeds and cross-sectional areas. Associated with unequal areas are different loss coefficients across the junction depending on flow direction. Unequal pipe section areas create a potential ambiguity in the loss coefficient

$$k = \frac{\Delta H}{V^2/(2g)} \tag{59}$$

because the fluid velocity V will be different in the pipe sections on the two sides of the junction. To resolve this ambiguity, we have simulation blocks with the loss coefficients defined in terms of the velocity in the downstream (closer to plant) and upstream (closer to the inlet) sides of the junction.

5.3.2 J-1: Junction with loss coefficients defined by downstream velocity head

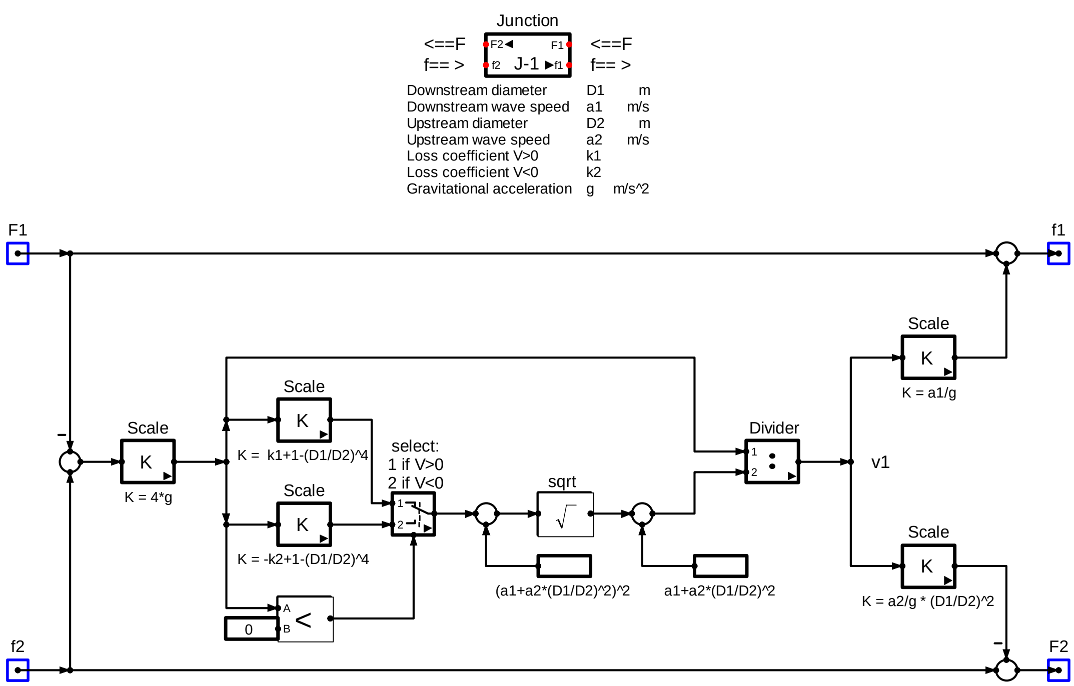

Figure 2: Block diagram of general pipe junction with loss coefficients defined by downstream velocity head.

Figure 2 shows the program block J-1 for a junction between dissimilar pipe sections. The block has seven parameters:

1. pipe section diameters D_1 and D_2 on sides 1 and 2,
2. wave speeds a_1, a_2 on sides 1 and 2,
3. loss coefficient k_1 for positive flow direction (2 to 1),
4. loss coefficient k_2 for negative flow direction (1 to 2), and
5. gravitational acceleration g.

In block J-1 the loss coefficients are defined by the downstream velocity head. Expressing the difference in total head loss as a fraction of the *downstream* velocity head $V_1^2/(2g)$, we have

$$\left(f_2 + F_2 + \frac{V_2^2}{2g}\right) - \left(f_1 + F_1 + \frac{V_1^2}{2g}\right) = k\frac{V_1^2}{2g} \tag{60}$$

where V_1 is the downstream fluid velocity, V_2 is the upstream fluid velocity, $k = k_1$ for $V_1 > 0$ and $k = -k_2$ for $V_1 < 0$. From Equation 57 we can express each of the *unknown* head waves

$$f_1 = F_1 + \frac{a_1}{g}V_1 \tag{61}$$

$$F_2 = f_2 - \frac{a_2}{g}V_2 \tag{62}$$

in terms of the *known* head waves f_2 and F_1 and the fluid velocities V_1 and V_2. Substituting these expressions for f_1 and F_2 into Equation 60 and rearranging, we obtain

$$4g\left(f_2 - F_1\right) = V_1\left(2a_1 + V_1\right) + V_2\left(2a_2 - V_2\right) + kV_1^2. \tag{63}$$

But from conservation of mass

$$V_2 = \left(\frac{D_1}{D_2}\right)^2 V_1 \tag{64}$$

and so

$$4g\left(f_2 - F_1\right) = \left(\left(2a_1 + V_1\right) + \left(\frac{D_1}{D_2}\right)^2\left(2a_2 - V_2\right) + kV_1\right)V_1. \tag{65}$$

From the definition of k, $kV_1 > 0$. So, in the case where the wave propagation speeds a_1 and a_2 are larger than the fluid speeds $|V_1|$ and $|V_2|$,

$$\left(2a_1 + V_1\right) + \left(\frac{D_1}{D_2}\right)^2\left(2a_2 - V_2\right) + kV_1 > 0. \tag{66}$$

From this it follows that V_1 has the same sign as $f_2 - F_1$. Eliminating the remaining upstream fluid velocity V_2 from Equation 65, we have

$$4g\left(f_2 - F_1\right) = \left(\left(2a_1 + V_1\right) + \left(\frac{D_1}{D_2}\right)^2 \left(2a_2 - \left(\frac{D_1}{D_2}\right)^2 V_1\right) + kV_1\right) V_1. \tag{67}$$

Examining the solutions to the last equation

$$V_1 = \frac{-\left(a_1 + \left(\frac{D_1}{D_2}\right)^2 a_2\right) \pm \sqrt{\left(a_1 + \left(\frac{D_1}{D_2}\right)^2 a_2\right)^2 + 4g\left(f_2 - F_1\right)\left(k - \left(\frac{D_1}{D_2}\right)^4 + 1\right)}}{k - \left(\frac{D_1}{D_2}\right)^4 + 1}, \tag{68}$$

it can be seen that the sign of V_1 is the same as the sign of $f_2 - F_1$ only when the *positive* square root is selected. In this case, the numerator is the difference between nearly equal quantities. Loss of numerical precision is avoided in the simulation program by using the equivalent formula

$$V_1 = \frac{4g\left(f_2 - F_1\right)}{\left(a_1 + \left(\frac{D_1}{D_2}\right)^2 a_2\right) + \sqrt{\left(a_1 + \left(\frac{D_1}{D_2}\right)^2 a_2\right)^2 + 4g\left(f_2 - F_1\right)\left(k - \left(\frac{D_1}{D_2}\right)^4 + 1\right)}}. \tag{69}$$

The upstream velocity V_2 is then computed using Equation 64 and the block outputs f_1 and F_2 are computed using Equations 61 and 62.

5.3.3 J-2: Junction with loss coefficients defined by upstream velocity head

Figure 3 shows the program block J-2 for a junction between dissimilar pipe sections. The parameters are the same as for block J-1 except that in block J-2 the loss coefficients k_1 and k_2 are defined in terms of the upstream velocity head. Expressing the difference in total head loss as a fraction of the *upstream* velocity head, we have

$$\left(f_2 + F_2 + \frac{V_2^2}{2g}\right) - \left(f_1 + F_1 + \frac{V_1^2}{2g}\right) = k\frac{V_2\left|V_2\right|}{2g} \tag{70}$$

where $k = k_1$ for $V_2 > 0$ and $k = -k_2$ for $V_2 < 0$. This equation can be reduced to an equation with the upstream velocity V_2 being the only unknown by means of Equations 61 and 62 together with

$$V_1 = \left(\frac{D_2}{D_1}\right)^2 V_2. \tag{71}$$

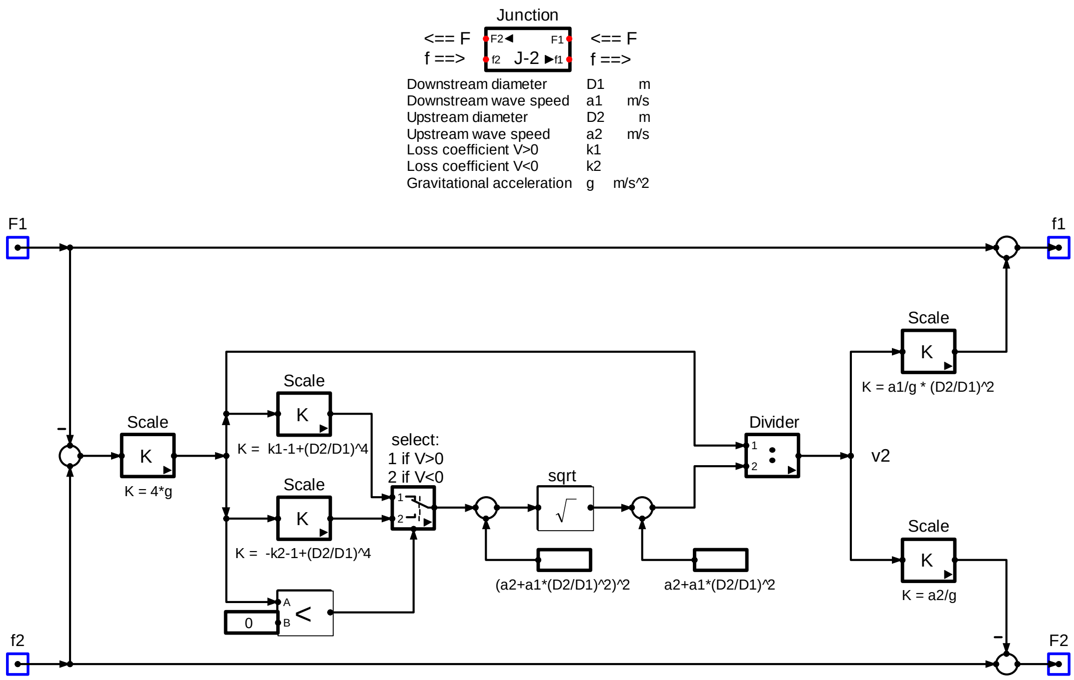

Figure 3: Block diagram of pipe junction with losses defined by upstream velocity head.

In block J-2 we compute the upstream velocity using the formula

$$V_2 = \frac{4g\left(f_2 - F_1\right)}{\left(a_2 + \left(\frac{D_2}{D_1}\right)^2 a_1\right) + \sqrt{\left(a_2 + \left(\frac{D_2}{D_1}\right)^2 a_1\right)^2 + 4g\left(f_2 - F_1\right)\left(k + \left(\frac{D_2}{D_1}\right)^4 - 1\right)}}. \tag{72}$$

The downstream velocity V_1 is then computed using Equation 71 and the block outputs f_1 and F_2 are computed using Equations 61 and 62.

5.3.4 E-1: Pipe entrance

Figure 4 shows program block E-1 for a pipe entrance. This is a limiting case of the program block J-1 (Section 5.3.2) in which the upstream diameter D_2 goes to infinity. Equation 69 then reduces to

$$V_1 = \frac{4g\left(f_2 - F_1\right)}{a_1 + \sqrt{a_1^2 + 4g\left(f_2 - F_1\right)\left(k + 1\right)}} \tag{73}$$

where $k = k_1$ for $V_1 > 0$ and $k = -k_2$ for $V_1 < 0$. Equation 64 reduces to $V_2 = 0$ and Equation 62 reduces to

$$F_2 = f_2. \tag{74}$$

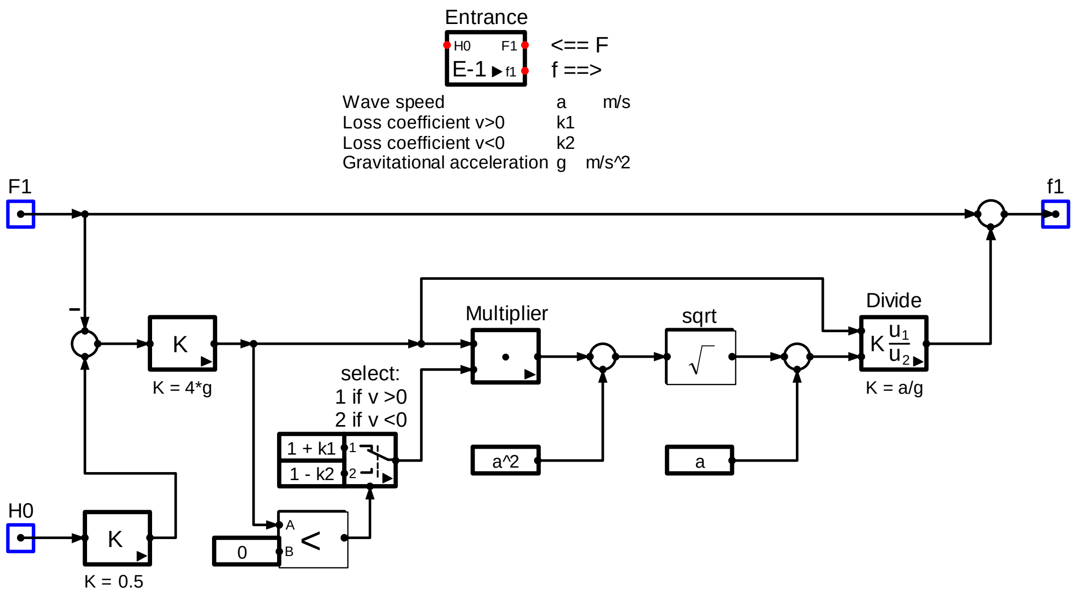

Figure 4: Block diagram of pipe entrance.

And since the hydraulic head upstream of the pipe entrance is

$$H_0 = f_2 + F_2, \tag{75}$$

we must have

$$f_2 = H_0/2. \tag{76}$$

The fluid velocity downstream of the intake is then

$$V_1 = \frac{4g\,(H_0/2 - F_1)}{a_1 + \sqrt{a_1^2 + 4g\,(H_0/2 - F_1)\,(k+1)}} \tag{77}$$

and we can compute the head wave f_1 travelling downstream from the intake using Equation 61.

5.3.5 D-1: Pipe discharge

Figure 5 shows program block D-1 for a pipe discharge. This is a limiting case of the general junction J-2 (Section 5.3.3) in which the downstream diameter D_1 goes to infinity. Equation 72 then reduces to

$$V_2 = \frac{4g\,(f_2 - F_1)}{a_2 + \sqrt{a_2^2 + 4g\,(f_2 - F_1)\,(k-1)}} \tag{78}$$

where $k = k_1$ for $V_2 > 0$ and $k = -k_2$ for $V_2 < 0$. Equation 71 reduces to $V_1 = 0$ and Equation 61 reduces to

$$f_1 = F_1. \tag{79}$$

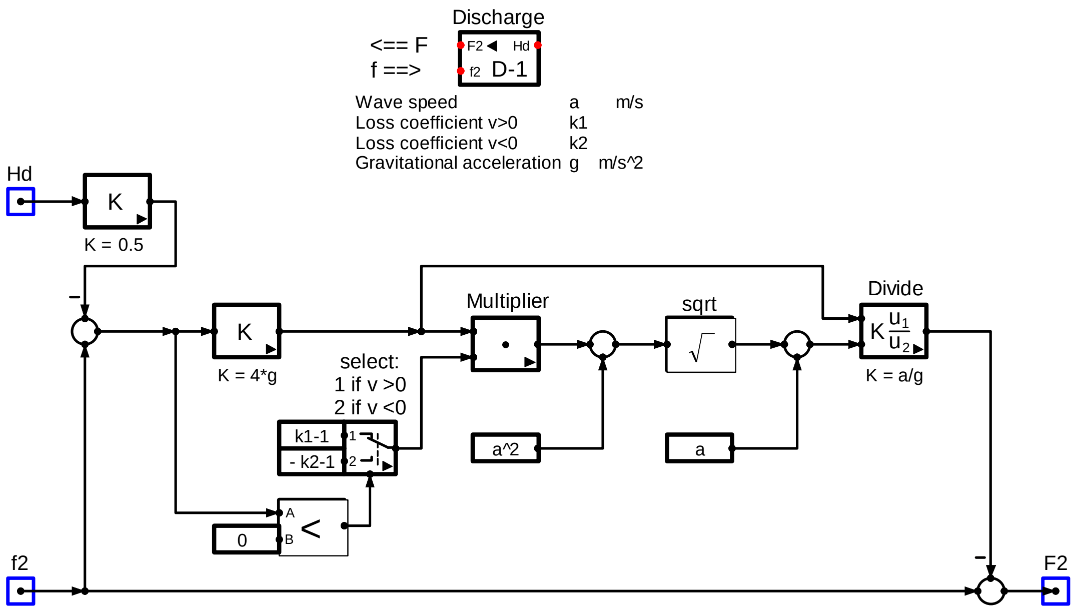

Figure 5: Block diagram for pipe discharge.

And since the hydraulic head downstream of the pipe discharge is

$$H_d = f_1 + F_1, \tag{80}$$

we must have

$$F_1 = H_d/2. \tag{81}$$

The fluid velocity upstream of the discharge is then

$$V_2 = \frac{4g\,(f_2 - H_d/2)}{a_2 + \sqrt{a_2^2 + 4g\,(f_2 - H_d/2)\,(k-1)}} \tag{82}$$

and we can compute the head wave F_2 travelling upstream from the discharge using Equation 62.

5.4 Junctions Between Similar Pipe Sections

5.4.1 L-1: Loss junction between similar pipe sections

Figure 6 shows program block L-1 for a junction between similar pipe sections. This is a special case of the general junctions J-1 and J-2 in which the pipe sections on each side of the junction have the same diameter D and wave speed a and where the loss coefficients k do not depend on the flow direction. In this special case, Equations

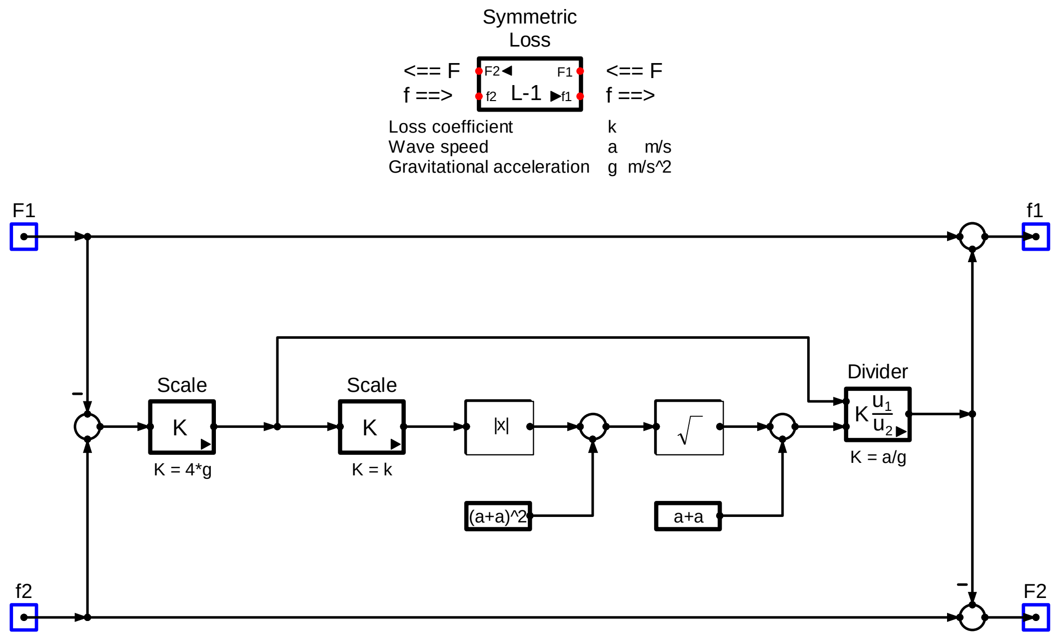

Figure 6: Block diagram of constant symmetrical loss in uniform pipe.

69 and 72 each reduce to

$$V = \frac{4g\,(f_2 - F_1)}{2a + \sqrt{(2a)^2 + 4gk\,|f_2 - F_1|}} \tag{83}$$

and block outputs f_1 and F_2 are computed using Equations 61 and 62.

5.4.2 W-1: Pipe section with friction loss

Figure 7 shows the program block W-1 for a pipe section with friction loss. The distributed friction loss is approximated by a single loss concentrated at the middle of the pipe section. This is accomplished by sandwiching a symmetric loss block L-1 between two elementary pipe sections W-0. Each of the elementary pipe sections has a travel time which is half the travel time for the entire pipe section. The loss coefficient is specified by

$$k = \frac{fL}{D} \tag{84}$$

where f is the Darcy-Weisbach friction factor, L is the entire pipe section length, and D is the hydraulic diameter of the pipe. The hydraulic diameter is defined to be

$$D = \frac{4A}{P} \tag{85}$$

where A is the cross-sectional area and P is the wetted perimeter of the cross-section. For round pipes, the hydraulic diameter is the same as the inside diameter.

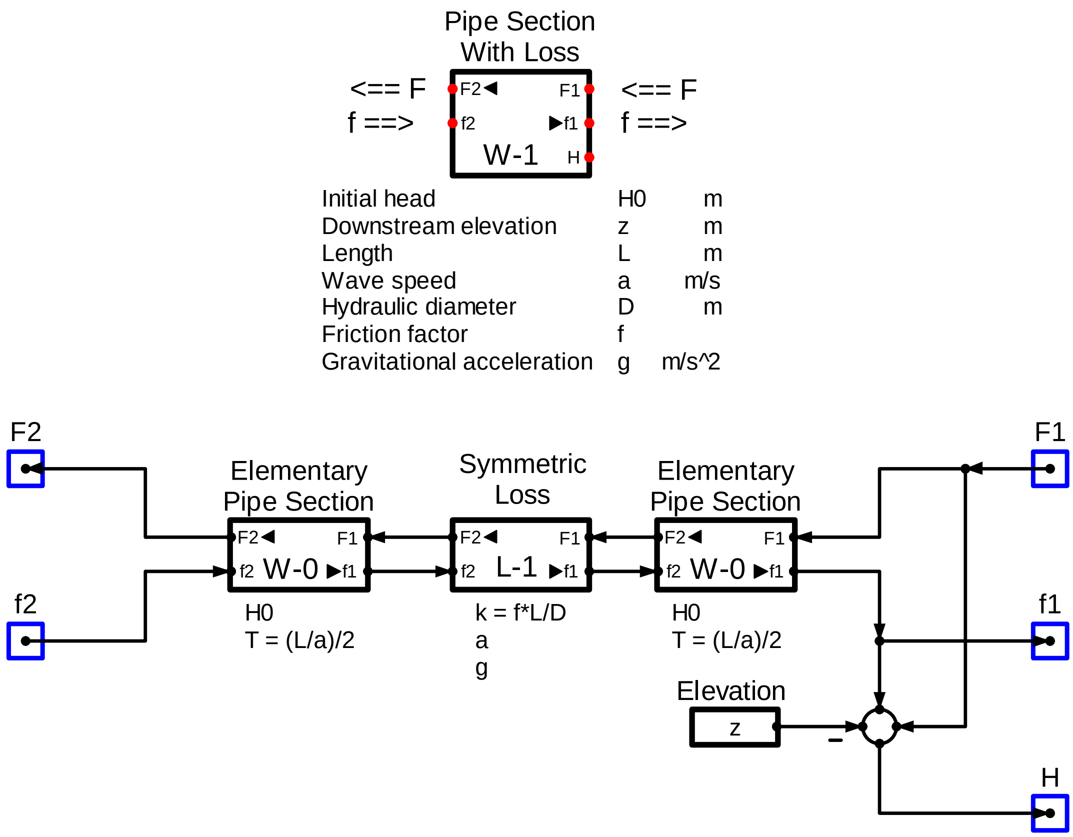

Figure 7: Block diagram of pipe section with loss.

The static head H at side 1 (the downstream end for normal flow direction) is calculated according to

$$H = (f_1 + F_1) - z \tag{86}$$

where z is the elevation of the downstream end of the pipe section. The static head H is the static gauge pressure expressed in meters of water.

5.4.3 V-1: In-line valve with variable loss coefficient

Figure 8 shows the program block V-1 for a symmetric loss junction with a *variable* loss coefficient k. This block is used to represent valves. The loss calculation in V-1 is optimized for large loss coefficients up to and including infinity. For this reason the valve opening is specified by the reciprocal of the loss coefficient $1/k$. An input of $1/k = 0$ corresponds to a fully closed valve. Rewriting Equation 68 in the symmetrical case with equal upstream and downstream fluid velocities $V = V_1 = V_2$,

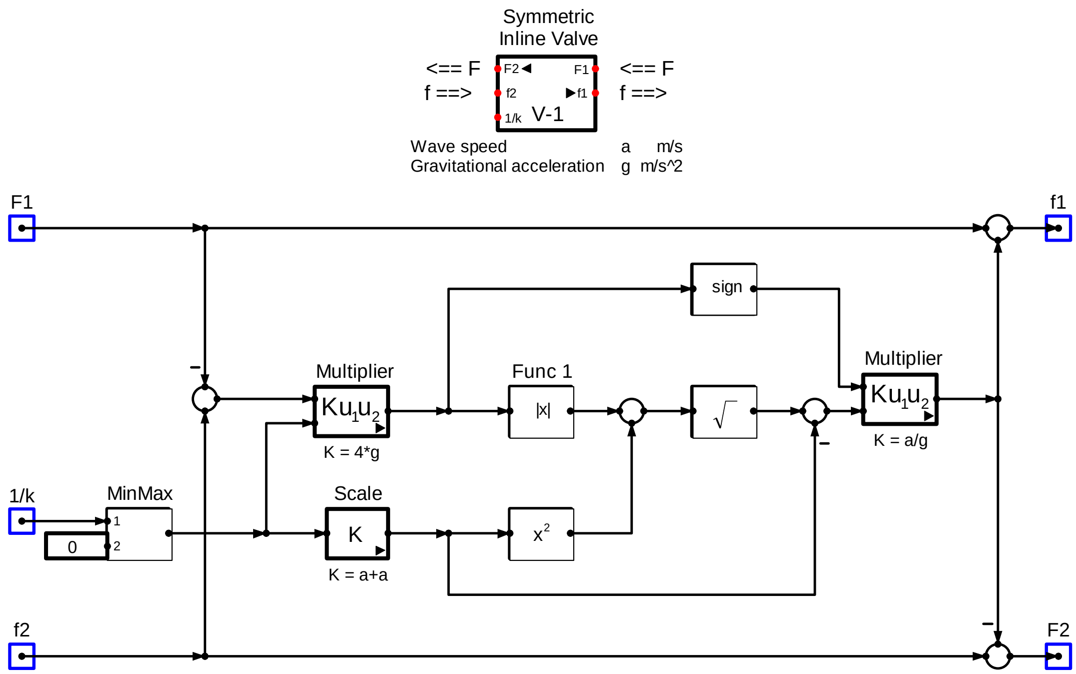

Figure 8: Block diagram for inline valve with variable loss coefficient.

pipe diameters $D = D_1 = D_2$ and wave speeds $a = a_1 = a_2$, we obtain

$$V = \frac{-(2a) + \sqrt{(2a)^2 + 4g\,(f_2 - F_1)\,k'}}{k'}, \tag{87}$$

where $k' = k$ if $f_2 - F_1 \geq 0$ and $k' = -k$ if $f_2 - F_1 < 0$. Block V-1 computes the fluid velocity on both sides of the valve using the equivalent formula

$$V = \text{sign}\,(f_2 - F_1)\left[\sqrt{(2a\,(1/k))^2 + 4g\,|f_2 - F_1|\,(1/k)} - 2a\,(1/k)\right]. \tag{88}$$

5.4.4 V-2: In-line valve with variable flow coefficient

Figure 9 shows the program block V-2 in which the valve is characterized by the flow coefficient

$$C_V = \frac{Q}{\sqrt{\Delta H}} \tag{89}$$

where Q is the flow through the valve and ΔH is the head drop across the valve. The program block computes the equivalent reciprocal of the loss coefficient

$$\frac{1}{k} = \frac{1}{\Delta H}\frac{V^2}{2g} = \frac{C_V^2}{Q^2}\frac{V^2}{2g} = \frac{1}{2g}\frac{V^2}{Q^2}C_V^2 = \frac{1}{2gA^2}C_V^2 \tag{90}$$

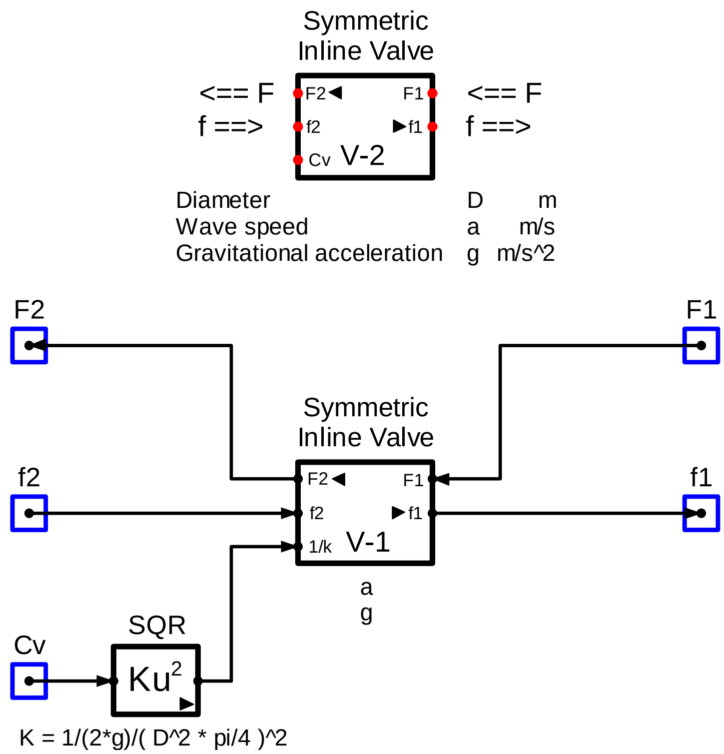

Figure 9: Block diagram for inline valve.with variable flow coefficient.

where the pipe cross-sectional area A is computed from the pipe diameter according to

$$A = \frac{\pi}{4} D^2. \tag{91}$$

The wave equations are then solved using block V-1.

5.5 B-1: Bifurcation

Figure 10 shows the program block B-1 for a pipe bifurcation with no losses. The bifurcation is required to have equal total head in each of the three branches

$$f_3 + F_3 + \frac{V_3^2}{2g} = f_1 + F_1 + \frac{V_1^2}{2g} = f_2 + F_2 + \frac{V_2^2}{2g}. \tag{92}$$

Also the flow into branch 3 must equal the sum of the flows exiting branches 1 and 2, thus

$$D_3^2 V_3 = D_1^2 V_1 + D_2^2 V_2. \tag{93}$$

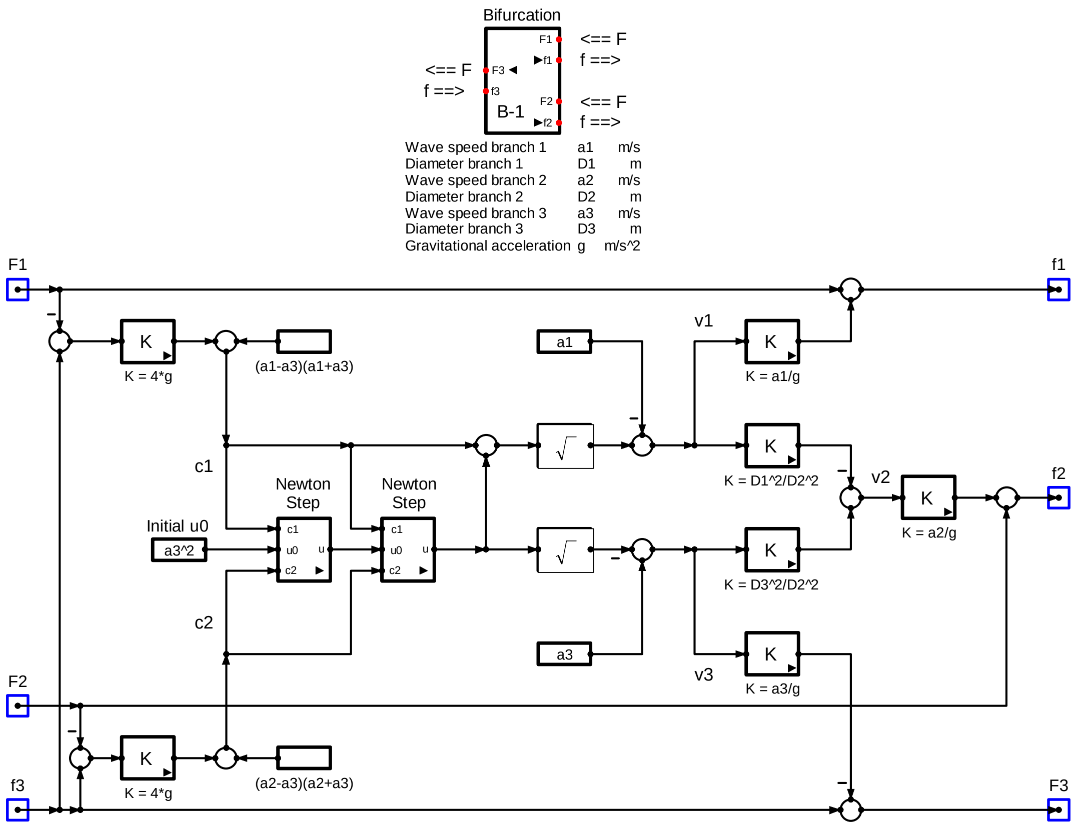

Figure 10: Block diagram of bifurcation with no loss.

As for the previous junctions, the *unknown* head waves f_1, f_2, and F_3 can be expressed in terms of the velocities and the *known* head waves F_1, F_2, and f_3 according to

$$f_1 = F_1 + \frac{a_1}{g} V_1 \tag{94}$$

$$f_2 = F_2 + \frac{a_2}{g} V_2 \tag{95}$$

$$F_3 = f_3 - \frac{a_3}{g} V_3. \tag{96}$$

Inserting these expressions into Equation 92 and simplifying, we obtain

$$V_3^2 - 2a_3 V_3 + 4g f_3 = V_1^2 + 2a_1 V_1 + 4g F_1 = V_2^2 + 2a_2 V_2 + 4g F_2. \tag{97}$$

The direct solution for the three velocities V_1, V_2, and V_3 from Equations 93 and 97 would require solving a fourth degree polynomial. Instead, we reduce these two equations to a single nonlinear equation in one unknown which we solve using an iterative numerical scheme.

The nonlinear equation is derived by first completing the squares in Equation 97

$$(a_3 - V_3)^2 + \left(-a_3^2 + 4g f_3\right) = (a_1 + V_1)^2 + \left(-a_1^2 + 4g F_1\right) = (a_2 + V_2)^2 + \left(-a_2^2 + 4g F_2\right) \tag{98}$$

and making a change of variables

$$u = (a_3 - V_3)^2\,, \quad u_1 = (a_1 + V_1)^2\,, \quad u_2 = (a_2 + V_2)^2\,. \tag{99}$$

This change of variables yields the equivalent linear system

$$u + \left(-a_3^2 + 4gf_3\right) = u_1 + \left(-a_1^2 + 4gF_1\right) = u_2 + \left(-a_2^2 + 4gF_2\right). \tag{100}$$

From this linear system we can express

$$u_1 = u + c_1$$

and

$$u_2 = u + c_2$$

in terms of the single variable u and two constants

$$c_1 = a_1^2 - a_3^2 + 4g\left(f_3 - F_1\right) \tag{101}$$

and

$$c_2 = a_2^2 - a_3^2 + 4g\left(f_3 - F_2\right). \tag{102}$$

We can now express the three unknown fluid velocities in terms of the single unknown u according to

$$V_1 = \sqrt{u_1} - a_1 = \sqrt{u + c_1} - a_1 \tag{103}$$
$$V_2 = \sqrt{u_2} - a_2 = \sqrt{u + c_2} - a_2 \tag{104}$$
$$V_3 = a_3 - \sqrt{u}. \tag{105}$$

Inserting these expressions for the fluid velocities into Equation 93 we obtain the requirement that

$$0 = G\left(u\right) = D_1^2\left(\sqrt{u + c_1} - a_1\right) + D_2^2\left(\sqrt{u + c_2} - a_2\right) + D_3^2\left(\sqrt{u} - a_3\right) \tag{106}$$

where $G\left(u\right)$ is a nonlinear function of the one unknown u. Program block B-1 finds an approximate solution to this equation by Newton's method. In this method we approximate $G\left(u\right)$ near an approximate solution u_0 by

$$G\left(u\right) \approx G\left(u_0\right) + \left(u - u_0\right) G'\left(u_0\right). \tag{107}$$

The value of u for which the approximation of $G\left(u\right)$ is zero

$$u = u_0 - \frac{G\left(u_0\right)}{G'\left(u_0\right)} \tag{108}$$

will be closer to the true solution of $G\left(u\right) = 0$ than the original approximation u_0. Figure 11 shows the program block for computing the improved approximation of u

$$u = u_0 - \frac{D_1^2\sqrt{u_0 + c_1} + D_2^2\sqrt{u_0 + c_2} + D_3^2\sqrt{u_0} - \left(a_1 D_1^2 + a_2 D_2^2 + a_3 D_3^2\right)}{\frac{1}{2}\left(D_1^2/\sqrt{u_0 + c_1} + D_2^2/\sqrt{u_0 + c_2} + D_3^2/\sqrt{u_0}\right)} \tag{109}$$

given an approximation u_0. The bifurcation block uses a two-step Newton approximation to calculate an approximate value of u. The fluid velocities V_1 and V_3 are then computed using Equations 103 and 105, and fluid velocity V_2 is computed directly from Equation 93. Finally the output head waves f_1, f_2, and F_3 are computed using Equations 94, 95, and 96.

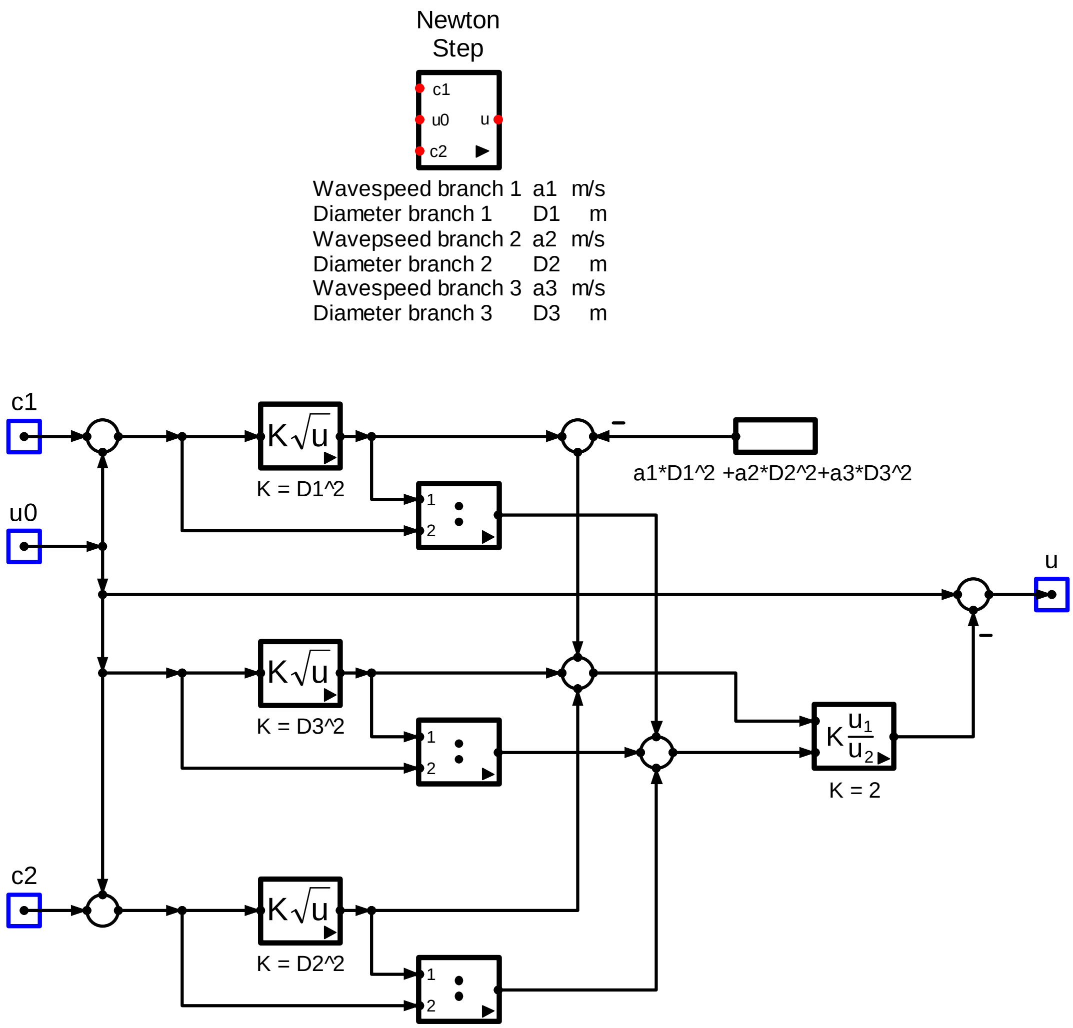

Figure 11: Details of single Newton step for solving nonlinear equation $G(u) = 0$.

6 Area Modulus Formulas

For an axially constrained tube, the displacement dr of the wall material at a radius r is given by

$$dr = \frac{(1+\nu)}{E\left(r_o^2 - r_i^2\right)}\left[\left((1-2\nu)r_i^2 r + \frac{r_i^2 r_o^2}{r}\right)dp_i - \left((1-2\nu)r_o^2 r + \frac{r_i^2 r_o^2}{r}\right)dp_o\right] \quad (110)$$

where dp_i and dp_o are the changes in the inside and outside pressures respectively and r_i and r_o are the initial inside and outside radii.[1] For thin-walled tubes, where $r_0 - r_i$ is much smaller than r_0, we have the approximation

$$dr = \frac{1-\nu^2}{E}\frac{r^2}{r_o - r_i}\left(dp_i - dp_o\right). \quad (111)$$

For constant outside pressure p_o, the area modulus is therefore

$$K_1 = A\frac{dp_i}{dA} = \frac{r_i}{2}\frac{dp_i}{dr_i} = \frac{Ee}{D\left(1-\nu^2\right)} \tag{112}$$

where D is the tube diameter and e is the wall thickness.

For very thick tubes (i.e. tunnels) we have the approximation

$$dr = \frac{(1+\nu)}{E}\left[\frac{r_i^2}{r}dp_i - \left((1-2\nu)r + \frac{r_i^2}{r}\right)dp_o\right] \tag{113}$$

for r and r_i very small compared to r_0. From this it follows that the area modulus is

$$K_2 = \frac{r_i}{2}\frac{dp_i}{dr_i} = \frac{E}{2\left(1+\nu\right)} = G. \tag{114}$$

In the case of a lined tunnel we have a thin-walled tube bonded to a very thick tube each with different material properties. Letting b subscript denote where the materials are bonded. We then have following variables:

1. inside radius of pipe r_i,
2. inside pressure of pipe p_i,
3. outside radius of pipe r_b,
4. outside pressure of pipe p_b,
5. inside radius of tunnel r_b,
6. inside pressure of tunnel p_b, and
7. outside pressure of tunnel p_o assumed constant.

From Equation 111 we have for the pipe

$$dr_i = \frac{1-\nu^2}{E}\frac{r_i^2}{r_b - r_i}\left(dp_i - dp_b\right) \tag{115}$$

and from Equation 113 we have for the tunnel

$$dr_b = \frac{(1+\nu)}{E}r_b dp_b. \tag{116}$$

We then have an area modulus of

$$\begin{aligned} K_3 &= \frac{r_i}{2}\frac{dp_i}{dr_i} \\ &= \frac{r_i}{2}\left(\frac{d(p_i - p_b)}{dr_i} + \frac{dp_b}{dr_i}\right) \\ &\approx \frac{r_i}{2}\frac{d(p_i - p_b)}{dr_i} + \frac{r_b}{2}\frac{dp_b}{dr_b} \\ &= K_1 + K_2. \end{aligned} \tag{117}$$

where we used the approximations $r_i \approx r_b$ and $dr_i \approx dr_b$ consistent with thin-walled approximation for the steel pipe.

7 References

[1] *Mechanics of Materials*, Egor P. Popov, Second edition, Prentice-Hall, Inc., 1978.